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Lemniscates and K-spectral sets

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Abstract

We show how multicentric representation of functions provides a simple way to generalize the von Neumann result that the unit disc is a spectral set for contractions in Hilbert spaces. In particular the sets need not be connected and the results can be applied to bounding Riesz spectral projections.

Keywords: von Neumann spectral sets; K-spectral sets; Lemniscates; Multicentric representation; Jacobi series; Riesz spectral projections

1. Multicentric representation of holomorphic functions

1.1. Motivation

If A denotes a bounded operator in a Hilbert space, we denote

$$V_p(A) = \{z \in \mathbb{C}\colon \left|p(z)\right| \leqslant \left\|p(A)\right\|\} \tag{1.1}$$

where p is a monic polynomial with distinct roots. We shall show that these sets are K-spectral, whenever the lemniscate does not pass through any critical point of p. As any compact set can be the spectrum of a bounded operator, it is crucial that lemniscates do have good approximation properties, a topic which was started by D. Hilbert in 1897, see e.g. [11]. Furthermore, we have recently provided an algorithm [5] which, for any given bounded A, produces a sequence

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of monic polynomials p with distinct roots such that the sets $V_p(A)$ squeeze around the polynomially convex hull of the spectrum of A.

In order to be able to estimate a holomorphic function f effectively at an operator A we use the approach we introduced in [6], which one may view as a combination of *Jacobi series* [4,11] with *Lagrange interpolation*. Each monic polynomial

$$p(z) = \prod_{j=1}^{d} (z - \lambda_j)$$

with simple roots λ_j induces a unique *multicentric representation* of f,

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(w), \quad \text{with } w = p(z). \tag{1.2}$$

Here δ_k denote the polynomials of degree $d-1$ taking the value 1 at λ_k and vanishing at the other roots. In [6] we have also discussed the practical computation of the Taylor series of f_k. In fact, the coefficients can be computed in a recursive fashion if the derivatives of the original function f are available at the local centers λ_k.

1.2. Key estimate

The representation (1.2) allows an obvious avenue for analysis, estimation and computation in complicated sets. One just treats the functions f_k in discs $|w| \leqslant R$ and combines the estimates for f in the sets satisfying $|p(z)| \leqslant R$.

In this paper we demonstrate this approach by generalizing a well-known result of von Neumann on contractions in Hilbert spaces. In order to do this we need to have an estimate of the following form

$$\sup_{|w| \leqslant R} \left| f_k(w) \right| \leqslant C(R) \sup_{|p(z)| \leqslant R} \left| f(z) \right|. \tag{1.3}$$

Such an estimate would then imply that the sets $V_p(A)$ are *K-spectral sets* with some K. In order to state it we need some notation. Let γ_R denote the lemniscate

$$\gamma_R = \left\{ z \in \mathbb{C}\colon \left| p(z) \right| = R \right\}.$$

For small R the lemniscate consists of d separate circular curves, for large R it reduces to just one circular curve. In general the lemniscate is smooth except if it contains a critical point, where the derivative of p vanishes. Thus there are at most $d-1$ such exceptional values R. Let $s(R)$ denote the distance from γ_R to the set of critical points.

Theorem 1.1. *If p is a monic polynomial of degree d with distinct roots, then there exists a constant C such that if f is holomorphic for $|p(z)| \leqslant R$, then the functions f_k in* (1.2) *are* *holomorphic for* $|w| \leqslant R$ *and if* γ_R *does not contain any critical points of* p *the estimate* (1.3) *holds with some* $C(R)$ *satisfying*

$$C(R) \leqslant 1 + \frac{C}{s(R)^{d-1}}. \tag{1.4}$$

Remark 1.2. If $C(R)$ denotes the smallest constant such that (1.3) holds for all f then $C(R) \to 1$ as $R \to 0$ or $R \to \infty$. Generically the critical points are simple and then the constant is proportional to $1/s(R)$ but we include an example where the behavior is of the form $1/s(R)^{d-1}$.

We postpone the proof while first giving applications to the spectral set theory.

2. Applications to *K*-spectral sets

2.1. *K-spectral sets using the von Neumann theorem*

We recall the definitions related to this topic. We denote by $B(H)$ the space of bounded linear operators in a Hilbert space H.

Definition 2.1. A closed set $\Sigma \subset \mathbb{C}$ is a spectral set for $A \in B(H)$, if for all rational functions r with poles off Σ there holds

$$\|r(A)\| \leqslant \sup_{z \in \Sigma} |r(z)|. \tag{2.1}$$

If the equation holds in the form

$$\|r(A)\| \leqslant K \sup_{z \in \Sigma} |r(z)|,$$

with a fixed K, then Σ is called a K-spectral set.

The topic began with a fundamental result by von Neumann for contractions in Hilbert spaces.

Theorem 2.2. *(See von Neumann (1951) [10].) If* $A \in B(H)$*, and* $\|A\| \leqslant 1$*, then the closed unit disc is a spectral set for* A.

This can clearly be reformulated also as follows:

$$\|f(A)\| \leqslant \sup_{|z| \leqslant \|A\|} |f(z)| \tag{2.2}$$

provided f is holomorphic in $|z| \leqslant \|A\|$.

We formulate our results for holomorphic functions rather than for polynomials or rational functions as we consider sets which may consist of several simply connected components. In particular, then the results apply as such for Riesz spectral projections. Here is the main result of this paper.

Theorem 2.3. *Suppose we are given a monic $p \in \mathbb{P}_d$ with distinct roots and a bounded operator $A \in B(H)$ in a Hilbert space H. Let $R \geqslant 0$ satisfy $\|p(A)\| \leqslant R$ and be such that the lemniscate γ_R contains no critical points of p. Then for all f which are holomorphic for $|p(z)| \leqslant R$ there holds*

$$\|f(A)\| \leqslant K \sup_{|p(z)|\leqslant R} |f(z)|, \tag{2.3}$$

where the constant K satisfies

$$K \leqslant C(R) \sum_{k=1}^{d} \|\delta_k(A)\|, \tag{2.4}$$

with $C(R)$ as in Theorem 1.1.

Proof. The claim follows immediately from Theorem 1.1 and from the von Neumann Theorem 2.2. In fact, denoting $B = p(A)$ we have from (2.2)

$$\|f_k(B)\| \leqslant \sup_{|w|\leqslant R} |f_k(w)|$$

and so by Theorem 1.1

$$\|f_k(B)\| \leqslant C(R) \sup_{|p(z)|\leqslant R} |f(z)|.$$

Then the result follows from

$$f(A) = \sum_{k=1}^{d} \delta_k(A) f_k(B). \qquad \square$$

2.2. Application to the Riesz spectral projections

A simple but useful application of the previous result is obtained as follows. Suppose γ_R consists of several components and is free from critical points. Then one can define f to be identically 1 in some open neighborhood of some of the components and to vanish in a neighborhood of all the others. If $A \in B(H)$ is such that $\|p(A)\| \leqslant R$, then the resulting operator is simply the *Riesz spectral projection* to the invariant subspace w.r.t. the part of the spectrum where f equals 1.

The following example shows that the constant $C(R)$ of Theorem 1.1 has to blow up near the critical lemniscates, and that the worst behavior in (1.4) may happen.

Example 2.4. Let $\varepsilon > 0$ be small. Consider the matrix

$$A(\varepsilon) = \begin{pmatrix} \varepsilon & 1 \\ 0 & -\varepsilon \end{pmatrix}, \tag{2.5}$$

with spectrum $\sigma(A(\varepsilon)) = \{\varepsilon, -\varepsilon\}$. Let $p(\lambda) = \lambda^2 - 1$ so that we have one critical point at the origin. Put $R = 1 - \varepsilon^2$ so that the spectrum lies on the boundary of the lemniscate $|p(z)| = \|p(A(\varepsilon))\| = 1 - \varepsilon^2$. Let f be 1 on the right open half plane and 0 on the left open half plane, so in particular it is holomorphic inside and in a neighborhood of the lemniscate. Then $f(A(\varepsilon)) = P(\varepsilon)$ is well defined and equals the *Riesz spectral projection* onto the direction of the eigenvector w.r.t. the eigenvalue ε. In fact, the resolvent satisfies

$$\big(\lambda I - A(\varepsilon)\big)^{-1} = \frac{1}{\lambda^2 - \varepsilon^2}\begin{pmatrix} \lambda + \varepsilon & 1 \\ 0 & \lambda - \varepsilon \end{pmatrix}.$$

The Riesz projector can be obtained as the residue at ε:

$$P(\varepsilon) = \begin{pmatrix} 1 & 1/2\varepsilon \\ 0 & 0 \end{pmatrix}.$$

As the distance from γ_R to the critical point is ε, we have

$$\|P(\varepsilon)\| \sim \frac{1/2}{s(R)}.$$

Likewise, if $p(\lambda) = \lambda^d - 1$ we could take $R = 1 - \varepsilon^d$ and e.g. the truncated backward shift and perturb it slightly:

$$S(\varepsilon) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 0 & 1 \\ \varepsilon^d & 0 & \cdot & \cdot & 0 \end{pmatrix}.$$

Again, the eigenvalues are at distance ε from the origin and the projection to the direction of an eigenvector behaves like

$$\|P(\varepsilon)\| \sim \frac{1/d}{s(R)^{d-1}}.$$

In fact,

$$P(\varepsilon) = \frac{1}{d\varepsilon^{d-1}}\begin{pmatrix} \varepsilon^{d-1} & \varepsilon^{d-2} & \cdot & \cdot & 1 \\ \varepsilon^d & \varepsilon^{d-1} & \cdot & \cdot & \varepsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varepsilon^{2d-2} & \varepsilon^{2d-3} & \cdot & \cdot & \varepsilon^{d-1} \end{pmatrix}.$$

In this case we would take the analytic function f to be identically 1 in the component which contains the point 1 and in the others we set it equal to zero. Then again $f(S(\varepsilon)) = P(\varepsilon)$.

2.3. Application to power boundedness

We can apply Theorem 2.3 with $R = 0$ but then A has to be an algebraic operator and all eigenvalues are nondefective. Requiring $p(A) = 0$ with p simple zeros says exactly that.

Consider now *power bounded* operators. Let $\mathbb{D}$ denote the open unit disc. Suppose that

$$V_p(A) \subset \overline{\mathbb{D}}. \tag{2.6}$$

If $p(\lambda) = (\lambda - 1)^2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then $p(A) = 0$, $V_p(A) = \{1\} \subset \overline{\mathbb{D}}$ but A is not power bounded as

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Observe that if p has multiple zeros, then $w = 0$ is a critical value. Thus we need to exclude this case.

Corollary 2.5. *Let p be monic with simple zeros and suppose $R \geqslant 0$ is such that γ_R contains no critical points and $\gamma_R \subset \overline{\mathbb{D}}$. If $A \in B(H)$ is such that $\|p(A)\| \leqslant R$, then A is power bounded and with the constant $C(R)$ provided by Theorem* 1.1 *we have for all $n \geqslant 1$,*

$$\|A^n\| \leqslant C(R) \sum_{k=1}^{d} \|\delta_k(A)\|. \tag{2.7}$$

Proof. The claim follows from

$$\max_{z \in \gamma_R} |z^n| \leqslant 1.$$

Notice that here the fact that the constant is larger than 1 is not important but instead we need that the *same* constant works for all holomorphic f. □

Remark 2.6. If A is *algebraic* then we can take p to be the *minimal polynomial*. Recall that a polynomial is called minimal polynomial if it is monic, $p(A) = 0$ and the polynomial is of smallest possible degree. Then Corollary 2.5 and Remark 1.2 yield

$$\|A^n\| \leqslant \sum_{k=1}^{d} \|\delta_k(A)\|. \tag{2.8}$$

Observe that this can be obtained directly as follows. For every n there exists a polynomial q_n such that

$$z^n = \sum_{k=1}^{d} \lambda_k^n \delta_k(z) + q_n(z) p(z)$$

which implies (2.8), as $p(A) = 0$.

Example 2.7. Let $B(\varepsilon) = \frac{1}{\varepsilon} S(\varepsilon)$ with $S(\varepsilon)$ as in Example 2.4. Then

$$B(\varepsilon)^d = I.$$

In particular, $B(\varepsilon)$ is power bounded: for $k \geqslant 0$ and $0 \leqslant m \leqslant d-1$,

$$\left\| B(\varepsilon)^{kd+m} \right\| = \frac{1}{\varepsilon^m}. \tag{2.9}$$

The minimal polynomial,

$$p(\lambda) = \lambda^d - 1$$

is independent of ε and so are the basis polynomials $\delta_k(z)$. As $\varepsilon \to 0$ we have

$$\left\| \delta_k\big(B(\varepsilon)\big) \right\| = \frac{1}{d\varepsilon^{d-1}}\big(1 + o(1)\big)$$

and

$$\sum_{k=1}^{d} \left\| \delta_k\big(B(\varepsilon)\big) \right\| = \frac{1}{\varepsilon^{d-1}}\big(1 + o(1)\big)$$

so that the bound in (2.8) gives

$$\left\| B(\varepsilon)^n \right\| \leqslant \frac{1}{\varepsilon^{d-1}}\big(1 + o(1)\big),$$

comparing well with (2.9).

2.4. Extensions based on the numerical range

Recall that the numerical range

$$W(A) = \big\{ (Ax, x) \in \mathbb{C}\colon\ x \in H,\ \|x\| = 1 \big\}$$

of $A \in B(H)$ is always convex, its closure contains the spectrum and it is included in the disc $|z| \leqslant \|A\|$. Denote by $w(A)$ the numerical radius of A:

$$w(A) = \sup_{z \in W(A)} |z|.$$

It is natural to ask whether the unit disc is a K-spectral set if $w(A) \leqslant 1$. Since for

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$w(A) = 1$ and $\|A\| = 2$, the constant K cannot in general be smaller than 2. K. Okubo and T. Andô [7] have shown, improving an earlier result in [9], that indeed, $K = 2$ always suffices. Based on this result we can formulate the following modification of Theorem 2.3.

Corollary 2.8. *If the assumptions of Theorem* 2.3 *hold with the only exception that the condition* $\|p(A)\| \leqslant R$ *is relaxed to* $w(p(A)) \leqslant R$, *then the conclusion holds with constant* K *satisfying*

$$K \leqslant 2C(R) \sum_{k=1}^{d} \|\delta_k(A)\|.$$

More recently B. Delyon and F. Delyon [3] have generalized Okubo's and Andô's result from the disc to arbitrary compact convex sets.

Theorem 2.9. *(See B. Delyon and F. Delyon [3].) If* Σ *is a convex compact set such that* $W(A) \subset \Sigma$ *then there exists a constant* K_Σ *such that*

$$\|R(A)\| \leqslant K_\Sigma \sup_{z \in \Sigma} |R(z)| \tag{2.10}$$

for all rational R *holomorphic in* Σ.

Remark 2.10. M. Crouzeix [2] has shown that $K_\Sigma \leqslant 11.08$. He conjectured that $K_\Sigma = 2$ will always do.

We can extend this result in the same way as that of Okubo and Andô. In fact, let $p \in \mathbb{P}_d$ be a monic polynomial with distinct zeros and $A \in B(H)$ be given. Put $B = p(A)$ and assume $W(B) \subset \Sigma$ with Σ convex and compact with a boundary that contains no critical points of p. Then we have

$$\|f_k(B)\| \leqslant K_\Sigma \sup_{w \in \Sigma} |f_k(w)| \tag{2.11}$$

Suppose we have an estimate of the form

$$\sup_{w \in \Sigma} |f_k(w)| \leqslant C_\Sigma \sup_{z \in p^{-1}(\Sigma)} |f(z)|. \tag{2.12}$$

Then we can combine these two inequalities as follows.

Theorem 2.11. *Suppose we are given a monic* $p \in \mathbb{P}_d$ *with distinct roots and a bounded operator* $A \in B(H)$ *in a Hilbert space* H. *Let* Σ *be a convex compact set such that* $W(p(A)) \subset \Sigma$

and such that the boundary of Σ contains no critical values of p. Then for all f which are holomorphic in $p^{-1}(\Sigma)$ we have

$$\|f(A)\| \leqslant C_\Sigma K_\Sigma \sum_{k=1}^{d} \|\delta_k(A)\| \sup_{z\in p^{-1}(\Sigma)} |f(z)|. \tag{2.13}$$

Proof. All we need is the existence of C_Σ in (2.12). We include this as a remark at the end of the proof of Theorem 1.1. □

Remark 2.12. Clearly many of the existing results invite generalizations of this nature. To name still one, in [1] the authors have considered operators such that several discs are simultaneously spectral sets and shown that their intersection is then K-spectral. Again, one could pose the assumptions on $p(A)$ and then consider the boundary of the intersection of the related discs. If the boundary is free from critical values, the preimage of the intersection is again K-spectral with some K.

Remark 2.13. Recall that an operator $A \in B(H)$ is called *polynomially bounded* if for some K,

$$\|p(A)\| \leqslant K \sup_{|z|\leqslant 1} |p(z)| \tag{2.14}$$

holds for all polynomials p. Such an operator is not necessarily similar to a contraction [8]. To guarantee it the inequality has to hold not only for scalar polynomials but for polynomials with matrix valued coefficients. Such operators are called *completely polynomially bounded* and it is clear by the proof technique of using the multicentric representation that the versions with matrix valued functions extend in the very same way as the scalar ones.

3. Proof of Theorem 1.1

In this section we discuss the estimation problem: if

$$f(z) = \sum_{k=1}^{d} \delta_k(z) f_k(w) \quad \text{where } w = p(z), \tag{3.1}$$

under what conditions we can bound f_k in terms of f. In particular, the discussion gives a proof for Theorem 1.1.

Proposition 3.1. *If f is holomorphic inside γ_{R_0}, then for $k = 1, \ldots, d$ the functions f_k are holomorphic for $|w| < R_0$.*

Proof. In [6] we discussed the multicentric representation by first writing the multicentric decomposition for the Cauchy kernel. This yields a separate kernel $K_k(z, w)$ for each center λ_k,

$$K_k(z, w) = \frac{1}{z - \lambda_k} \frac{p(z)}{p(z) - w}.$$

It then follows that for $|w| < R < R_0$ we have

$$f_k(w) = \frac{1}{2\pi i} \int_{\gamma_R} K_k(z, w) f(z)\, dz. \qquad \square \tag{3.2}$$

The integral representation (3.2) leads to bounds of the form

$$\sup_{|w| \leqslant r} \left| f_k(w) \right| \leqslant C(R, r) \sup_{|p(z)| \leqslant R} \left| f(z) \right|$$

with $r < R$ but here we are interested in having $r = R$. In order to prove such a bound we consider a different type of explicit representation for f_k in terms of f.

Assume first that w is a noncritical value of p and denote the d different roots of $p - w$ by $\zeta_j = \zeta_j(w)$,

$$p(\zeta_j) - w = 0$$

with numbering such that $\zeta_j(w) \to \lambda_j$ as $w \to 0$. Next, let $\varepsilon_j \in \mathbb{P}_{d-1}$ be the polynomials such that they equal 1 at ζ_j and vanish at the other roots ζ_l,

$$\varepsilon_j(\zeta_l) = \delta_{jl}. \tag{3.3}$$

The roots ζ_l are analytic functions of w away from the critical values, and so are the coefficients of ε_j, too.

Proposition 3.2. *If w is noncritical, then*

$$f_k(w) = \sum_{j=1}^{d} \varepsilon_j(\lambda_k) f\big(\zeta_j(w)\big). \tag{3.4}$$

Proof. For fixed w we introduce two polynomials, P and Q as follows

$$P(\zeta) = \sum_{k=1}^{d} \delta_k(\zeta) f_k(w)$$

while

$$Q(\zeta) = \sum_{j=1}^{d} \varepsilon_j(\zeta) f(\zeta_j).$$

By (3.1)

$$f(\zeta_j) = \sum_{k=1}^{d} \delta_k(\zeta_j) f_k(w) = P(\zeta_j)$$

and therefore

$$Q(\zeta) = \sum_{j=1}^{d} \varepsilon_j(\zeta) P(\zeta_j).$$

But then P and Q are the same polynomial. Substituting $\zeta = \lambda_k$ we obtain

$$Q(\lambda_k) = \sum_{j=1}^{d} \varepsilon_j(\lambda_k) f(\zeta_j) = P(\lambda_k) = f_k(w). \qquad \square$$

Assume now that f is holomorphic inside and on γ_R and let w_0 be a critical value of p such that $|w_0| < R$. For simplicity, let us assume that z_0 is the only critical point such that

$$p(z_0) = w_0.$$

The modifications needed with several such critical points are obvious and left to the reader. Assume that the roots $\zeta_j(w)$ are numbered such that for $j = 1, \ldots, m$ we have $\zeta_j(w) \to z_0$ as $w \to w_0$ while the other roots stay within a positive distance from the critical point. Then we may denote

$$\zeta_j(w) = z_0 + \bigl(\zeta_1(w) - z_0\bigr) e^{2\pi i \frac{j-1}{m}} \bigl(1 + o(1)\bigr). \tag{3.5}$$

By Proposition 3.1 we know that w_0 must be a removable singularity of

$$w \mapsto \sum_{j=1}^{m} \varepsilon_j(\lambda_k) f\bigl(\zeta_j(w)\bigr). \tag{3.6}$$

In order to compute the limit, we use the explicit representation of the polynomials:

Lemma 3.3. *At noncritical w we have*

$$\varepsilon_j(\zeta) = \frac{p(\zeta) - w}{p'(\zeta_j(w))(\zeta - \zeta_j(w))}.$$

In particular, as $p(\lambda_k) = 0$,

$$\varepsilon_j(\lambda_k) = \frac{w}{p'(\zeta_j(w))(\zeta_j(w) - \lambda_k)}.$$

Proof. This is just the usual formula for the basis polynomials in Lagrange interpolation, applied to the nodes $\zeta_l(w)$. $\square$

Hence the individual polynomials ε_j are not bounded as $\zeta_j(w) \to z_0$. However, we have with some $c \neq 0$,

$$p'\big(\zeta_j(w)\big) = \big(\zeta_j(w) - z_0\big)^{m-1} c + O\big(\big(\zeta_j(w) - z_0\big)^m\big),$$
$$w = w_0 + O\big(\big(\zeta_j(w) - z_0\big)^m\big),$$
$$\zeta_j(w) - \lambda_k = (z_0 - \lambda_k) + \big(\zeta_j(w) - z_0\big),$$

and after expanding $f(\zeta_j(w))$ into power series around z_0 and substituting all expansions into (3.6) we see that the key terms are of the form

$$\frac{const}{(\zeta_1(w) - z_0)^{m-1-n}} \sum_{j=1}^{m} e^{-2\pi i \frac{j-1}{m}(m-1-n)}$$

with $n \geqslant 0$. As long as $n < m - 1$ all sums vanish and hence (3.6) stays bounded as $w \to w_0$.

Suppose now that γ_R does not contain any critical points. Then for $|w| < R$,

$$\big|f_k(w)\big| \leqslant C_k(R) \sup_{z\in\gamma_R} \big|f(z)\big|$$

with

$$C_k(R) = \sup_{|w|=R} \sum_{j=1}^{d} \big|\varepsilon_j(\lambda_k)\big|. \tag{3.7}$$

What remains is to estimate $C(R)$ from (3.7).

From Lemma 3.3 we see that near w_0,

$$\big|\varepsilon_j(\lambda_k)\big| \sim \frac{|w_0|}{|z_0 - \lambda_k|} \frac{1}{s(R)^{m-1}}$$

where we denote by $s(R)$ the distance from γ_R to the nearest critical point. Further, as $R \to 0$ we have $\varepsilon_j(\lambda_k) \to \delta_{jk}$, and so $C_k(R) \to 1$. When $R \to \infty$, we have

$$\varepsilon_j(\lambda_k) = \frac{1 + o(1)}{d}$$

and again $C_k(R) \to 1$. Hence, there exists C such that for all $R \geqslant 0$,

$$C_k(R) \leqslant 1 + \frac{C}{s(R)^{d-1}}.$$

Naturally, generically the critical points are simple and in such a case one can bound $C_k(R)$ by $1/s(R)$.

Example 3.4. Let $p(z) = z^2 - 1$ and consider the lemniscate with $R = 1$ so that γ_1 passes through the origin which is a critical point of p. We show that there exists no finite $C(1)$ such that

$$\sup_{|z^2-1|\leqslant 1} \big|f_k\big(z^2 - 1\big)\big| \leqslant C(1) \sup_{|z^2-1|\leqslant 1} \big|f(z)\big| \tag{3.8}$$

would hold for all f holomorphic inside and on γ_1. One checks easily that if we set $\lambda_k = (-1)^{k+1}$, then

$$f_k\big(z^2-1\big) = \frac{1}{2}\big(f(z)+f(-z)\big) + (-1)^{k+1}\frac{1}{2z}\big(f(z)-f(-z)\big).$$

Consider the one-parameter family of functions with $\varepsilon > 0$:

$$f(z;\varepsilon) = \frac{\varepsilon}{z - i\varepsilon}.$$

Then we have, as $\varepsilon \to 0$,

$$\sup_{|z^2-1|\leqslant 1} \big|f_k\big(z^2-1;\varepsilon\big)\big| \sim \frac{const}{\varepsilon},$$

while

$$\sup_{|z^2-1|\leqslant 1} \big|f(z;\varepsilon)\big| = \mathcal{O}(1).$$

Example 3.5. We can use the previous example to also demonstrate what happens in estimating $f(A)$ if the lemniscate contains a critical point. Let $f(\varepsilon)$ and p be as in the previous example and take

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.9}$$

so that $p(A) = -I$. Then in particular $\|p(A)\| = 1$ and the eigenvalue of A is at the critical point. Again, as $\varepsilon \to 0$ we have

$$\sup_{|z^2-1|\leqslant 1} \big|f(z;\varepsilon)\big| = \mathcal{O}(1),$$

while

$$\big\|f(A;\varepsilon)\big\| \sim \frac{const}{\varepsilon}.$$

So, there exists no constant K such that for all $\varepsilon > 0$,

$$\big\|f(A;\varepsilon)\big\| \leqslant K \sup_{|p(z)|\leqslant 1} \big|f(z;\varepsilon)\big|.$$

In particular, the requirement, that γ_R contains no critical points, cannot be omitted in Theorem 2.3.

Remark 3.6. For the proof of Theorem 2.11 we need to establish the bound (2.12). However, all we need is to have a bound for

$$w \mapsto \sum_{j=1}^{d} \left| \varepsilon_j(\lambda_k) \right|$$

along the boundary of Σ. But as the boundary does not pass through any critical value, this function is continuous, and since the boundary is compact, the function is bounded.

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